Similarity transformation in one-dimensional reaction–diffusion systems: the voting model as an example

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Abstract. The exact solution for a system with two-particle annihilation and decoagulation has been studied. The spectrum of the Hamiltonian of the system is found. It is shown that the steady state is twofold degenerate. The average number density at each site $\langle n_i(t) \rangle$ and the equal-time two-point functions $\langle n_i(t)n_j(t) \rangle$ are calculated. Any equal-time correlation functions at large times, $\langle n_i(\infty)n_j(\infty) \cdots \rangle$, are also calculated. The relaxation behaviour of the system toward its final state is investigated and it is shown that generally it is exponential, as expected. For the special symmetric case, the relaxation behaviour of the system is a power law. For the asymmetric case, it is shown that the profile of deviation from the final values is an exponential function of the position.

1. Introduction

In recent years, reaction–diffusion systems have been studied by many people, using different methods. Among them are the field-theoretic methods, which allow for perturbative approaches to build up correlations in low dimensions [1, 2]. As mean-field techniques cannot be used for low-dimensional systems, people are motivated to study stochastic models in low dimensions, which can be solved exactly. Moreover, solving one-dimensional systems should, in principle, be easier. Applying a similarity transformation on an integrable model, one may construct stochastic models, their integrability may not be obvious. Recently, some people have studied such transformations [3–8].

Exact results for some models in a one-dimensional lattice have been obtained, for example, in [3, 9]. In these cases, the time evolution of the system is determined by a master equation [10]. Models with no diffusion received less attention in the literature [11–14]. It is said that unless the system has long-range reactions [12, 13], the time dependence involves exponential relaxation rather than power-law behaviour typical of the fast diffusion reactions.

In [15], a 10-parameter family of stochastic models has been studied. In these models, the k-point equal-time correlation functions $\langle n_i n_j \cdots n_k \rangle$ satisfy linear differential equations involving no higher-order correlations. These linear equations have been solved for the average density $\langle n_i \rangle$. However, this set of equations may not be solved easily for higher-order correlation functions. The spectrum is also partially obtained. The model which we address in this paper is a special case of that 10-parameter stochastic model.

In this work, we report the exact solution for a system with two-particle annihilation and decoagulation. This model may be considered as a biased voting model, in the sense that there

are two different opinions. If the two people on two adjacent sites have different opinions, they may interact so that their opinions become the same. The bias parameter corresponds to the dominance of the left (or right) site. In the absence of bias, this system is equivalent to the zero-temperature Glauber model [16, 17]. This system is related a free fermion system, through a similarity transformation, and hence is solvable. Note that the system itself is not a free fermion system and cannot be solved by only applying the Jordan–Wigner transformation.

When there is right–left symmetry, the average density decays to its final value in the form of a power law ($t^{-1/2}$). However, in the general case (a biased model) it decays in the form of an exponential. Moreover, the profile of the deviation of the average density from its final value is not uniform but exponential in terms of the site number. In fact, the parameter representing the right–left asymmetry, in some sense, determines the dominance of the right-hand sites over the left-hand sites, or vice versa.

The spectrum of the Hamiltonian of the system is found. It is shown that the steady state is twofold degenerate. The probability of finding the system in each of these two states is determined by the initial average density, and is time independent. It is shown that at large times, any n-point function is equal to the one-point function, which is position independent.

$$\langle n_i(\infty) n_j(\infty) \cdots n_k(\infty) \rangle = \langle n_i(\infty) \rangle = \frac{1}{L} \sum_m \langle n_m(0) \rangle. \tag{1}$$

This is due to the fact that the system has two steady states; either completely full, or completely empty, as will be shown. This means that the mean-field approach does not work and this system is highly correlated.

The paper is organized as follows. In section 2, similarity transformations relating stochastic systems to other (stochastic or non-stochastic) systems are investigated. In section 3, a solvable model is obtained through a similarity transformation on a free-fermion system. The spectrum of the system is also obtained in this section. In section 4, the one-point function is calculated and its large-time behaviour is investigated. In section 5, the two-point function and its limiting behaviour is obtained. In section 6, the null vectors of the Hamiltonian are obtained and from that the steady state of the system is obtained in terms of its one-point function at $t = 0$. Finally, in section 7 we consider the next-to-leading term of the one-point function at large times, and from this we obtain the way in which the system relaxes to its final state.

2. Similarity transformations as a method for obtaining solvable stochastic models

Here some standard material [2, 3, 5] is introduced, just to fix notation. The master equation for $P(\sigma, t)$ is

$$\frac{\partial}{\partial t} P(\sigma, t) = \sum_{\tau \neq \sigma} \left[\omega(\tau \to \sigma) P(\tau, t) - \omega(\sigma \to \tau) P(\sigma, t) \right] \tag{2}$$

where $\omega(\tau \to \sigma)$ is the transition rate from the configuration τ to σ. Introducing the state vector

$$|P(t)\rangle = \sum_\sigma P(\sigma, t) |\sigma\rangle \tag{3}$$

where the summation runs over all possible states of the system, one can write the above equation in the form

$$\frac{\partial}{\partial t} |P\rangle = \mathcal{H} |P\rangle \tag{4}$$

where the matrix elements of $\mathcal{H}$ are

$$\langle\sigma|\mathcal{H}|\tau\rangle = \omega(\tau\to\sigma) \qquad \tau\neq\sigma$$
$$\langle\sigma|\mathcal{H}|\sigma\rangle = -\sum_{\tau\neq\sigma}\omega(\sigma\to\tau). \tag{5}$$

The basis $\{\langle\sigma|\}$ is dual to $\{|\sigma\rangle\}$, that is

$$\langle\sigma|\tau\rangle = \delta_{\sigma,\tau}. \tag{6}$$

The operator is $\mathcal{H}$ is called a Hamiltonian, and it is not necessarily Hermitian. However, it has some properties. Conservation of probability,

$$\sum_{\sigma} P(\sigma,t) = 1 \tag{7}$$

shows that

$$\langle S|\mathcal{H} = 0 \tag{8}$$

where

$$\langle S| = \sum_{\beta}\langle\beta|. \tag{9}$$

So, the sum of each column of $\mathcal{H}$, as a matrix, should be zero. As $\langle S|$ is a left eigenvector of $\mathcal{H}$ with zero eigenvalue, $\mathcal{H}$ has at least one right eigenvector with zero eigenvalue. This state corresponds to the steady-state distribution of the system and it does not evolve in time. If the zero eigenvalue is degenerate, the steady state is not unique. The transition rates are non-negative, so the off-diagonal elements of the matrix $\mathcal{H}$ are non-negative. Therefore, if a matrix $\mathcal{H}$ has the following properties:

$$\langle S|\mathcal{H} = 0$$
$$\langle\sigma|\mathcal{H}|\tau\rangle \geqslant 0 \tag{10}$$

then it can be considered as the generator of a stochastic process. The real part of the eigenvalues of any matrix with the above conditions should be less than or equal to zero.

The dynamics of the state vectors (4) is given by

$$|P(t)\rangle = \exp(t\mathcal{H})|P(0)\rangle \tag{11}$$

and the expectation value of an observable $\mathcal{O}$ is

$$\langle\mathcal{O}\rangle(t) = \sum_{\sigma}\mathcal{O}(\sigma)P(\sigma,t) = \langle S|\mathcal{O}\exp(t\mathcal{H})|P(0)\rangle. \tag{12}$$

If $\mathcal{H}$ is integrable, one can solve the problem, that is, one can calculate the expectation values. Suppose now, that a Hamiltonian is integrable but is not stochastic. There arises a question as to whether or not there exists a similarity transformation which transforms it to a stochastic integrable Hamiltonian. Consider an integrable Hamiltonian $\tilde{\mathcal{H}}$. The similarity transformation

$$\mathcal{H} := \mathcal{B}\tilde{\mathcal{H}}\mathcal{B}^{-1} \tag{13}$$

leaves its eigenvalues invariant. Consider a special case: the system consists of a one-dimensional lattice, with nearest-neighbour interaction,

$$\tilde{\mathcal{H}} = \sum_{i=1}^{L}\tilde{\mathcal{H}}_{i,i+1}. \tag{14}$$

Suppose, also, that the system is translation invariant:

$$\tilde{\mathcal{H}}_{i\,i+1} = \underbrace{1 \otimes \cdots \otimes 1}_{i-1} \otimes \tilde{H} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L-i-1} \tag{15}$$

and we are using periodic boundary conditions. A simple class of similarity transformations is then

$$\mathcal{B} = \Gamma_1 \otimes \Gamma_2 \otimes \cdots \otimes \Gamma_L. \tag{16}$$

The simplest case is when all Γ_is are the same. In this case, if one can find Γ such that

$$H = \Gamma \otimes \Gamma \tilde{H} \Gamma^{-1} \otimes \Gamma^{-1} \tag{17}$$

is stochastic, then $\mathcal{H}$ defined through (13) would be stochastic. A more general class of similarity transformations is obtained through

$$\Gamma_i := \Gamma(g)^i \tag{18}$$

where g should have the property

$$[g \otimes g, \tilde{H}] = 0. \tag{19}$$

In this case, one obtains

$$H = (\Gamma \otimes \Gamma g)\tilde{H}(\Gamma \otimes \Gamma g)^{-1}. \tag{20}$$

Define $\langle s|$ to be the sum of all bra-states corresponding to a single site. We then have

$$\langle S| = \underbrace{\langle s| \otimes \cdots \otimes \langle s|}_{L}. \tag{21}$$

For H to be stochastic, its off-diagonal elements should be non-negative, and we must have

$$\langle s| \otimes \langle s|H = 0. \tag{22}$$

This shows that

$$\langle \alpha| \otimes \langle \beta| := \langle s|\Gamma \otimes \langle s|\Gamma g \tag{23}$$

should be an eigenvector with zero eigenvalue of $\tilde{H}$, that is, $\tilde{H}$ should have a decomposable left eigenvector. So, in order for this prescription of constructing an integrable stochastic model to work, one must begin with a Hamiltonian $\tilde{H}$, for which the left eigenvector with zero eigenvalue is decomposable. The real part of all other eigenvalues of $\tilde{H}$ should, of course, be non-positive.

3. A one-parameter solvable system on the basis of a free-fermion system

Consider the Hamiltonian

$$\tilde{\mathcal{H}} = \sum_{i=1}^{L} \left\{ \frac{1+\eta}{2}[s_{i+1}^+ s_i^- - n_i(1-n_{i+1})] + \frac{1-\eta}{2}[s_{i+1}^- s_i^+ - n_{i+1}(1-n_i)] \right.$$

$$\left. + \lambda[s_{i+1}^- s_i^- - n_i n_{i+1}] \right\} \tag{24}$$

where s^+, s^- and n are

$$s^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad s^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad n := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{25}$$

and the subscript i represents the site on which the operator acts. This Hamiltonian describes the following processes:

$$A\emptyset \to \emptyset A \qquad \text{with the rate } \frac{1+\eta}{2}$$

$$\emptyset A \to A\emptyset \qquad \text{with the rate } \frac{1-\eta}{2} \tag{26}$$

$$AA \to \emptyset\emptyset \qquad \text{with the rate } \lambda.$$
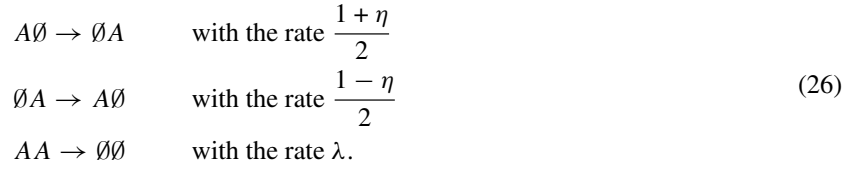
This model has been studied recently. In the case $\lambda = 0$, the above model describes an asymmetric exclusion process. For $\lambda = 1$, the Hamiltonian is bilinear in terms of creation s^+ and annihilation s^- operators. This problem has been solved via a Jordan–Wigner transformation [18, 19]. In the notation of the previous section the matrix form of $\tilde{H}$ is

$$\tilde{H} := \begin{pmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\frac{1}{2}(1+\eta) & \frac{1}{2}(1-\eta) & 0 \\ 0 & \frac{1}{2}(1+\eta) & -\frac{1}{2}(1-\eta) & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix}. \tag{27}$$

This matrix has two eigenvalues, 0 and -1, both of which are two-fold degenerate. One of the zero left eigenvectors can be decomposed into a tensor product. Performing the above-mentioned procedure, this Hamiltonian can be transformed into another stochastic one. For this case, one can show that the matrix g is the identity matrix, and the similarity transformation for all sites become the same. This has been done in [4].

One of the left eigenvectors corresponding to the eigenvalue -1 also has the desired property. To use the prescription described in the previous section to construct a stochastic Hamiltonian, we define a new Hamiltonian,

$$\tilde{H}' := -\tilde{H} - 1 \tag{28}$$

and apply the similarity transformation on this new Hamiltonian. One of the zero left eigenvectors of $\tilde{H}'$ is

$$(1 \quad 0 \quad 0 \quad 0) = (1 \quad 0) \otimes (1 \quad 0). \tag{29}$$

The similarity transformation should map $\langle s| \otimes \langle s|$ to this vector:

$$(1 \quad 1)\Gamma \otimes (1 \quad 1)\Gamma g = \alpha(1 \quad 0) \otimes (1 \quad 0). \tag{30}$$

So,

$$(1 \quad 1)\Gamma = \alpha\nu(1 \quad 0)$$
$$(1 \quad 1)\Gamma g = \frac{\alpha}{\nu}(1 \quad 0). \tag{31}$$

Scaling Γ and g does not alter the Hamiltonian H. So we can remove α and ν by scaling the matrices Γ and g. Then the above relation gives some constraints on the elements of Γ and g. The condition of positivity of rates, fixes g and Γ:

$$\Gamma = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{32}$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{33}$$

The two-site Hamiltonian, then, takes the following form:

$$H = \begin{pmatrix} 0 & \frac{1}{2}(1-\eta) & \frac{1}{2}(1+\eta) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{1}{2}(1+\eta) & \frac{1}{2}(1-\eta) & 0 \end{pmatrix} \tag{34}$$

and $\mathcal{H}$ is

$$\mathcal{H} = \sum_{i=1}^{L} \left\{ \frac{1-\eta}{2} \left[n_i s_{i+1}^+ + (1-n_i)s_{i+1}^- \right] + \frac{1+\eta}{2} \left[s_i^+ n_{i+1} + s_i^-(1-n_{i+1}) \right] \right.$$

$$\left. - \left[n_i(1-n_{i+1}) - (1-n_i)n_{i+1} \right] \right\}. \tag{35}$$

This Hamiltonian describes the following processes:

$$\begin{array}{ll} A\emptyset \to AA & \dfrac{1-\eta}{2} \\[2mm] A\emptyset \to \emptyset\emptyset & \dfrac{1+\eta}{2} \\[2mm] \emptyset A \to AA & \dfrac{1+\eta}{2} \\[2mm] \emptyset A \to \emptyset\emptyset & \dfrac{1-\eta}{2}. \end{array} \tag{36}$$

The Hamiltonian (35) is not quadratic in s^+ and s^-. So, one cannot map this Hamiltonian to a free fermion system, using a Jordan–Wigner transformation. However, the Hamiltonian $\tilde{\mathcal{H}}$ is integrable and can be mapped to a free fermion system by a Jordan–Wigner transformation. Consider the following Jordan–Wigner transformation [18, 19]:

$$\begin{aligned} a_j &:= Q_{j-1} s_j^- \\ a_j^\dagger &:= Q_{j-1} s_j^+ \\ Q_j &:= \prod_{i=1}^{j} (-s_i^3). \end{aligned} \tag{37}$$

It can be easily shown that the number operator at each site n_i is, in terms of new generators,

$$n_i := \frac{1+s_i^3}{2} = a_i^\dagger a_i. \tag{38}$$

Using this transformation, the Hamiltonian $\tilde{\mathcal{H}}$ takes the following form:

$$\tilde{\mathcal{H}} = \sum_{i=1}^{L} \left[\frac{1-\eta}{2} a_i^\dagger a_{i+1} + \frac{1+\eta}{2} a_{i+1}^\dagger a_i + a_{i+1}a_i - a_i^\dagger a_i \right]. \tag{39}$$

a_i and $a_i^\dagger$ fulfil the fermionic anti-commutation relations

$$\begin{aligned} \{a_i, a_j\} &= \{a_i^\dagger, a_j^\dagger\} = 0 \\ \{a_i, a_j^\dagger\} &= \delta_{ij}. \end{aligned} \tag{40}$$

Note that it is in the limit $L \to \infty$ that the Jordan–Wigner transformation we are using, works. Otherwise, there are some boundary terms in (39) as well. So, all the results we obtain hereafter, are valid only in this limit. Now, introducing the Fourier transformation

$$a_j := \frac{1}{\sqrt{L}} \sum_k b_k \exp\left\{\frac{2\pi \mathrm{i} j k}{L}\right\}$$

$$a_j^\dagger := \frac{1}{\sqrt{L}} \sum_k b_k^\dagger \exp\left\{\frac{-2\pi \mathrm{i} j k}{L}\right\}$$

(41)

and substituting it in (40), it is seen that

$$\{b_k, b_l\} = \{b_k^\dagger, b_l^\dagger\} = 0$$

$$\{b_k, b_l^\dagger\} = \delta_{kl}.$$

(42)

As a result, the Hamiltonian $\tilde{\mathcal{H}}$ takes the form

$$\tilde{\mathcal{H}} = \sum_k \left[\frac{1-\eta}{2} \exp\left(\frac{2\pi \mathrm{i} k}{L}\right) + \frac{1+\eta}{2} \exp\left(\frac{-2\pi \mathrm{i} k}{L}\right)\right] b_k^\dagger b_k + b_{-k} b_k \exp\left(\frac{-2\pi \mathrm{i} k}{L}\right) - b_k^\dagger b_k$$

$$= \sum_k \left[\epsilon_k b_k^\dagger b_k - \mathrm{i} \sin\left(\frac{2\pi k}{L}\right) b_{-k} b_k\right]$$

(43)

where

$$\epsilon_k := -1 + \cos\left(\frac{2\pi k}{L}\right) - \mathrm{i}\eta \sin\left(\frac{2\pi k}{L}\right).$$

(44)

One can now easily obtain the time dependence of b_k and $b_k^\dagger$, using (42) and $\mathrm{d}O/\mathrm{d}t = [O, H]$

$$b_k(t) = b_k(0) \, \mathrm{e}^{\epsilon_k t}$$

$$b_k^\dagger(t) = \mathrm{e}^{-\epsilon_k t}\left\{b_k^\dagger(0) - \mathrm{i} \cot\left(\frac{\pi k}{L}\right)\left[\mathrm{e}^{(\epsilon_k + \epsilon_{-k})t} - 1\right] b_{-k}(0)\right\}.$$

(45)

Now we return to our problem: determining the expectation values of a system evolving with the Hamiltonian $\mathcal{H}$. The expectation value of a quantity $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle(t) = \langle S|\mathcal{O} \exp(t\mathcal{H})|P(0)\rangle$$

$$= \langle S| \exp(-t\mathcal{H})\mathcal{O} \exp(\mathcal{H}t)|P(0)\rangle.$$

(46)

Substituting $\mathcal{H} = -\mathcal{B}\tilde{\mathcal{H}}\mathcal{B}^{-1} - L\mathbf{1}$, where $\mathbf{1}$ denotes the identity matrix, yields

$$\langle \mathcal{O} \rangle(t) = \langle \Omega|\tilde{\mathcal{O}}(-\tilde{t})\mathcal{B}^{-1}|P(0)\rangle$$

(47)

where

$$\tilde{\mathcal{O}} := \mathcal{B}^{-1}\mathcal{O}\mathcal{B}$$

(48)

$$\tilde{\mathcal{O}}(-\tilde{t}) := \mathrm{e}^{t\tilde{\mathcal{H}}}\mathcal{O}\mathrm{e}^{-t\tilde{\mathcal{H}}}$$

(49)

and

$$\langle \Omega| := \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \cdots \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

(50)

The main expectation values of interest are the correlation functions of n_i. To determine these, we use

$$\Gamma^{-1}n\Gamma = \tfrac{1}{2}(1 - s^+ - s^-)$$

$$(\Gamma g)^{-1}n\Gamma g = \tfrac{1}{2}(1 + s^+ + s^-).$$

(51)

So

$$\mathcal{B}^{-1} n_i \mathcal{B} = \tfrac{1}{2}[1 - (-1)^i (s^+ + s^-)]. \tag{52}$$

Now, we want to calculate the expectation value of $\mathcal{O}$ where

$$\mathcal{O} := n_{i_m} \cdots n_{i_2} n_{i_1} \qquad i_1 < i_2 < \cdots < i_m. \tag{53}$$

We have

$$\tilde{\mathcal{O}} = \frac{1}{2^m} \big[1 - (-1)^{i_m} (s_{i_m}^+ + s_{i_m}^-)\big] \cdots \big[1 - (-1)^{i_1} (s_{i_1}^+ + s_{i_1}^-)\big]. \tag{54}$$

Using the Jordan–Wigner transformation, one arrives at

$$\tilde{\mathcal{O}} = \frac{1}{2^m} \big[1 - (-1)^{i_m} Q_{i_m - 1} (a_{i_m}^\dagger + a_{i_m})\big] \cdots \big[1 - (-1)^{i_1} Q_{i_1 - 1} (a_{i_1}^\dagger + a_{i_1})\big]. \tag{55}$$

It is easy to check that $\langle \Omega | Q_i = (-1)^i \langle \Omega |$. So in calculating $\langle \mathcal{O} \rangle$, one can use $\mathcal{O}'$ instead of $\tilde{\mathcal{O}}$:

$$\mathcal{O}' := \frac{1}{2^m} \big[1 + (a_{i_m}^\dagger + a_{i_m})\big] \cdots \big[1 + (a_{i_1}^\dagger + a_{i_1})\big]. \tag{56}$$

Instead of $\mathcal{O}'$, It is enough to set $\mathcal{O}''$ in the expectation value of $\mathcal{O}$, where

$$\mathcal{O}'' := \frac{1}{2^m} \big(1 + a_{i_m}^\dagger\big) \cdots \big(1 + a_{i_1}^\dagger\big). \tag{57}$$

To prove this, one should use $\langle \Omega | \tilde{\mathcal{H}} = -L \langle \Omega |$ and $\langle \Omega | a_i(0) = 0$.

4. The one-point function

As the first example, consider the one-point function $\langle n_m(t) \rangle$:

$$\langle n_m(t) \rangle = \tfrac{1}{2} \langle \Omega | [1 + a_m^\dagger(-\tilde{t})] \mathcal{B}^{-1} | P(0) \rangle. \tag{58}$$

Using the Fourier transformation (41), the time dependence of $b_k^\dagger$ (45) and remembering $\langle \Omega | b_k(0) = 0$, we obtain

$$\langle n_m(t) \rangle = \frac{1}{2} + \frac{1}{2\sqrt{L}} \sum_k e^{-2\pi i k m / L} \langle \Omega | b_k^\dagger(0) \mathcal{B}^{-1} | P(0) \rangle e^{\epsilon_k t}. \tag{59}$$

Now, we use the inverse Fourier and Jordan–Wigner transformations, and arrive at

$$\langle n_m(t) \rangle = \frac{1}{2} + \frac{1}{2L} \sum_{k,j} e^{2\pi i k(j-m)/L} \langle S | \mathcal{B} s_j^+ \mathcal{B}^{-1} | P(0) \rangle e^{\epsilon_k t} (-1)^{j-1}. \tag{60}$$

This can be written in a simpler form, using

$$\mathcal{B} s_j^+ \mathcal{B}^{-1} = (-1)^{j-1} \frac{2 n_j - 1 + s_j^- - s_j^+}{2} \tag{61}$$

and

$$\langle s | (2n - 1) = \langle s | (s^- - s^+). \tag{62}$$

One then arrives at

$$\langle n_m(t) \rangle = \frac{1}{2} + \frac{1}{2L} \sum_{k,j} e^{2\pi i k(j-m)/L} \langle S | (2 n_j(0) - 1) | P(0) \rangle e^{\epsilon_k t}. \tag{63}$$

Using $\langle S|P(0)\rangle = \sum_\sigma P(\sigma, 0) = 1$, one arrives at

$$\langle n_m(t)\rangle = \sum_j \Lambda_{mj}(t)\langle n_j(0)\rangle \tag{64}$$

where

$$\Lambda_{mj}(t) := \frac{1}{L}\sum_k e^{2\pi ik(j-m)/L}e^{\epsilon_k t}. \tag{65}$$

Now, consider the limit $t \to \infty$. In this limit, the only contribution in the above summation comes from the term $k = 0$. So,

$$\lim_{t\to\infty}\langle n_m(t)\rangle = \frac{1}{L}\sum_j\langle n_j(0)\rangle \tag{66}$$

which shows that in the limit $t \to \infty$, the expectation value of the number of particles in any site tends to the average of the initial value of this quantity. In the final section, we will find the next leading term of $\langle n_i(t)\rangle$, for large times.

Now, we want to calculate the expectation value of the number of particles at the site j in the limit $L \to \infty$. First, we calculate Λ_{mj} in this limit. To do so, we define $z := \exp(i2\pi k/L)$. We then (in this limit) arrive at

$$\Lambda_{mj}(t) = e^{-t}\oint \frac{dz}{2\pi iz}z^{j-m}\exp\left[t\left(\tfrac{1}{2}(1-\eta)z + \tfrac{1}{2}(1+\eta)z^{-1}\right)\right]. \tag{67}$$

Changing the variable z to $w := z\sqrt{\frac{1+\eta}{1-\eta}}$, the above integral takes the form

$$\Lambda_{mj}(t) = e^{-t}\left(\frac{1-\eta}{1+\eta}\right)^{(j-m)/2}\oint\frac{dw}{2\pi iw}w^{j-m}\exp\left[t\sqrt{\frac{1-\eta^2}{2}}(w+w^{-1})\right] \tag{68}$$

or, using the change of variable $w := e^{i\theta}$,

$$\Lambda_{mj}(t) = e^{-t}\left(\frac{1-\eta}{1+\eta}\right)^{(m-j)/2}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{i(j-m)\theta+t\sqrt{1-\eta^2}\cos\theta}. \tag{69}$$

The above integral is an integral representation of the modified Bessel function:

$$\Lambda_{mj}(t) = \left(\frac{1-\eta}{1+\eta}\right)^{(m-j)/2}I_{m-j}(t\sqrt{1-\eta^2})e^{-t}. \tag{70}$$

$\langle n_m(t)\rangle$, in the limit $L \to \infty$, is then

$$\langle n_m(t)\rangle = \sum_j\left(\frac{1-\eta}{1+\eta}\right)^{(m-j)/2}I_{m-j}(t\sqrt{1-\eta^2})e^{-t}\langle n_j(0)\rangle. \tag{71}$$

5. The two-point function

The other quantity which we want to calculate is $\langle n_m(t)n_l(t)\rangle$. Without loss of generality, one may assume $(m > l)$. To calculate this, we use (57), which gives

$$\langle n_m(t)n_l(t)\rangle = \tfrac{1}{4}\langle\Omega|[1 + a_m^\dagger(-\tilde{t})][1 + a_l^\dagger(-\tilde{t})]\mathcal{B}^{-1}|P(0)\rangle$$

$$= -\tfrac{1}{4} + \tfrac{1}{2}[\langle n_m(t)\rangle + \langle n_l(t)\rangle] + \tfrac{1}{4}\langle\Omega|a_m^\dagger(-\tilde{t})a_l^\dagger(-\tilde{t})\mathcal{B}^{-1}|P(0)\rangle. \tag{72}$$

The main thing is to calculate the last term. To do this, we first use the Fourier transformation of $a_i^\dagger$s,

$$\langle\Omega|a_m^\dagger(-\tilde{t})a_l^\dagger(-\tilde{t}))\mathcal{B}^{-1}|P(0)\rangle = \frac{1}{L}\sum_{k,p} \mathrm{e}^{-\mathrm{i}2\pi(km+pl)/L}\langle\Omega|b_k^\dagger(-\tilde{t})b_p^\dagger(-\tilde{t})\mathcal{B}^{-1}|P(0)\rangle \tag{73}$$

and then substitute the time dependence of $b_k^\dagger$s.

$$\langle\Omega|a_m^\dagger(-\tilde{t})a_l^\dagger(-\tilde{t}))\mathcal{B}^{-1}|P(0)\rangle = \frac{1}{L}\sum_{k,p} \mathrm{e}^{-\mathrm{i}2\pi(km+pl)/L+(\epsilon_k+\epsilon_p)t}\langle\Omega|b_k^\dagger(0)$$

$$\times \left[b_p^\dagger(0) + \mathrm{i}\cot\left(\frac{\pi p}{L}\right)\left(1 - \mathrm{e}^{-(\epsilon_p+\epsilon_{-p})t}\right)b_{-p}(0)\right]\mathcal{B}^{-1}|P(0)\rangle. \tag{74}$$

Now we use the inverse Fourier transformation for the $b_k^\dagger b_p^\dagger$ term. The other term is easily summed. We arrive at,

$$\langle\Omega|a_m^\dagger(-\tilde{t})a_l^\dagger(-\tilde{t})\mathcal{B}^{-1}|P(0)\rangle = \sum_{r,s}\Lambda_{mr}(t)\Lambda_{ls}(t)\langle\Omega|a_r^\dagger a_s^\dagger\mathcal{B}^{-1}|P(0)\rangle$$

$$+\frac{\mathrm{i}}{L}\sum_k \mathrm{e}^{\mathrm{i}2\pi k(l-m)/L}\cot\left(\frac{\pi k}{L}\right)\left(1 - \mathrm{e}^{(\epsilon_k+\epsilon_{-k})t}\right) \tag{75}$$

or,

$$\langle\Omega|a_m^\dagger(-\tilde{t})a_l^\dagger(-\tilde{t})\mathcal{B}^{-1}|P(0)\rangle = \sum_{k,p,r,s}\Lambda_{mr}(t)\Lambda_{ls}(t)\langle(2n_r-1)(2n_s-1)\rangle_0\,\mathrm{sgn}(r-s)$$

$$+\frac{\mathrm{i}}{L}\sum_k \mathrm{e}^{\mathrm{i}2\pi k(l-m)/L}\cot\left(\frac{\pi k}{L}\right)\left(1 - \mathrm{e}^{(\epsilon_k+\epsilon_{-k})t}\right) \tag{76}$$

where we have used the definition of Λ_{ij}, and $\langle\cdots\rangle_0$ means the expectation value at the initial time. Adding all terms in (72) together, one arrives at

$$\langle n_m(t)n_l(t)\rangle = -\frac{1}{4}(\langle n_m(t)\rangle + \langle n_l(t)\rangle) + \sum_{r,s}\Lambda_{mr}(t)\Lambda_{ls}(t)\langle(n_r-\tfrac{1}{2})(n_s-\tfrac{1}{2})\rangle_0\,\mathrm{sgn}(r-s)$$

$$+\frac{\mathrm{i}}{4L}\sum_k \mathrm{e}^{\mathrm{i}2\pi k(l-m)/L}\cot\left(\frac{\pi k}{L}\right)\left(1 - \mathrm{e}^{(\epsilon_k+\epsilon_{-k})t}\right). \tag{77}$$

The last term is independent of the initial conditions. So we can calculate it for a special case, e.g. $|P(0)\rangle = |0\rangle$. Then, the final result is

$$\langle n_m(t)n_l(t)\rangle = \frac{1}{2}(\langle n_m(t)\rangle + \langle n_l(t)\rangle) + \sum_{r,s}\Lambda_{mr}(t)\Lambda_{ls}(t)\,\mathrm{sgn}(r-s)\langle n_r n_s - \tfrac{1}{2}(n_r+n_s)\rangle_0. \tag{78}$$

For large times, it is seen that

$$\lim_{t\to\infty}\langle n_m(t)n_l(t)\rangle = \langle n(\infty)\rangle. \tag{79}$$

6. Null vectors of the Hamiltonian, the steady state of the system and the n-point function

Now we want to study the null eigenvectors of the Hamiltonian $\mathcal{H}$. It is easy to see that the Hamiltonian (35) has at least two null eigenvectors, which means that the steady state is not unique. These states are one in which all sites are occupied, and one in which no site is

occupied. One can check this easily by acting the Hamiltonian (35) on these states. It was shown that the Hamiltonian $\tilde{\mathcal{H}}$ may be written as

$$\tilde{\mathcal{H}} = \sum_k \left[\epsilon_k b_k^\dagger b_k - \mathrm{i} \sin\left(\frac{2\pi k}{L}\right) b_{-k} b_k \right]$$

$$= \sum_{k>0} \left[\epsilon_k b_k^\dagger b_k + \epsilon_{-k} b_{-k}^\dagger b_{-k} - 2\mathrm{i} \sin\left(\frac{2\pi k}{L}\right) b_{-k} b_k \right] + \epsilon_0 b_0^\dagger b_0. \tag{80}$$

This Hamiltonian is obviously block diagonal. In each four-dimensional block, one can choose a basis $\{|0\rangle, b_k^\dagger|0\rangle, b_{-k}^\dagger|0\rangle, b_k^\dagger b_{-k}^\dagger|0\rangle\}$. The eigenvalues of this four-dimensional block are $0, \epsilon_k, \epsilon_{-k}, \epsilon_k + \epsilon_{-k}$ or $\mathcal{N}_k \epsilon_k + \mathcal{N}_{-k} \epsilon_{-k}$, where the $\mathcal{N}$s are zero or one. The eigenvalues of the Hamiltonian are, therefore,

$$\tilde{E}\{\mathcal{N}\} = \sum_k \mathcal{N}_k \epsilon_k. \tag{81}$$

From this, one can obtain the eigenvalues of $\mathcal{H}$ as

$$E\{\mathcal{N}\} = -\sum_k (\mathcal{N}_k \epsilon_k + 1)$$

$$= \sum_k (-\mathcal{N}_k + 1)\epsilon_k. \tag{82}$$

Here we have used

$$\sum_k (1 + \epsilon_k) = 0. \tag{83}$$

Now, it is easy to see that E is zero iff $1 - \mathcal{N}_k = 0$, $\forall k \neq 0$. This shows that the null eigenvector is twofold degenerate. As the final state is twofold degenerate, and it is known that the totally full state $\left(|\Omega\rangle := \binom{1}{0} \otimes \binom{1}{0} \otimes \cdots \binom{1}{0}\right)$ and totally empty state $\left(|0\rangle := \binom{0}{1} \otimes \binom{0}{1} \otimes \cdots \binom{0}{1}\right)$ are null eigenvectors of the system, we have

$$|P(\infty)\rangle = \alpha |0\rangle + \beta |\Omega\rangle \tag{84}$$

where $\alpha + \beta = 1$. Using (66), and

$$\langle S|n_i|0\rangle = 0 \qquad \langle S|n_i|\Omega\rangle = 1 \tag{85}$$

it is seen that

$$\beta = \langle n(\infty)\rangle = \frac{1}{L} \sum_j \langle n_j(0)\rangle =: \rho_0 \tag{86}$$

and

$$\alpha = 1 - \rho_0. \tag{87}$$

From this, one obtains

$$|P(\infty)\rangle = [1 - \rho_0]|0\rangle + \rho_0 |\Omega\rangle. \tag{88}$$

Using this, it is easy to find all m-point functions in the limit $t \to \infty$. We have

$$\langle n_{i_m} \cdots n_{i_1}\rangle(\infty) = \langle S|n_{i_m} \cdots n_{i_1}|P(\infty)\rangle$$

$$= \rho_0$$

$$= \langle n(\infty)\rangle$$

$$\neq \langle n(\infty)\rangle^m. \tag{89}$$

This clearly shows that the mean-field approximation does not work here.

7. Relaxation of the system toward its steady state

It was shown that in the limit $t \to \infty$, the expectation of the number of particles in any site tends to the average of the initial value of this quantity. Now, we want to study the behaviour of the system at large times. Starting from (71), and representing $\langle n_j(0) \rangle$ by its Fourier transform, we have

$$\langle n_m(t) \rangle = \mathrm{e}^{-t} \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \sum_j \left(\frac{1-\eta}{1+\eta} \right)^{(m-j)/2} I_{m-j}(t\sqrt{1-\eta^2}) \, \mathrm{e}^{-\mathrm{i}uj} [f(u) + 2\pi \bar{n} \delta(u)] \qquad (90)$$

where $[f(u) + 2\pi \bar{n} \delta(u)]$ is the Fourier transform of $\langle n_j(0) \rangle$ and $\bar{n}$ is the average density. We have extracted this part of the Fourier transform, so that the remaining is a smooth function of u. Then, $f(u)$ denotes the Fourier transform of the deviation $\langle n_j(0) \rangle - \bar{n}$. The summation on j is easily done, using

$$\sum_n x^n I_n(y) = \mathrm{e}^{(y/2)(x+1/x)}. \qquad (91)$$

So, one arrives at

$$\langle n_m(t) \rangle = \bar{n} + \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \mathrm{e}^{t\epsilon(u)} f(u) \, \mathrm{e}^{-\mathrm{i}mu} \qquad (92)$$

where $\epsilon(u)$ is the same as ϵ_k with $k = Lu/(2\pi)$. The above integral is simplified for large times, using the steepest-descent method. Using the change of variable $z := \mathrm{e}^{\mathrm{i}u}$, the integral becomes

$$\langle n_m(t) \rangle = \bar{n} + \oint \frac{\mathrm{d}z}{2\pi \mathrm{i}z} \mathrm{e}^{t[-2+(1-\eta)z+(1+\eta)z^{-1}]/2} \tilde{f}(z) z^{-m} \qquad (93)$$

where the integration contour is the unit circle. The multiplier of t in the exponent is stationary at

$$z_1 = \sqrt{\frac{1+\eta}{1-\eta}} \qquad (94)$$

and

$$z_2 = -\sqrt{\frac{1+\eta}{1-\eta}}. \qquad (95)$$

As the real part of this multiplier is larger at $z = z_1$, the integral gets its main contribution from this point. (This point is not on the integration contour. However, assuming $\tilde{f}$ to be analytic, one deforms the integration contour so that it passes from z_1, and then uses the steepest-descent method.) We arrive at

$$\langle n_m(t) \rangle - \bar{n} \sim \frac{1}{\sqrt{t}} \left(\frac{1-\eta}{1+\eta} \right)^{m/2} \mathrm{e}^{t(\sqrt{1-\eta^2}-1)}. \qquad (96)$$

The effect of the Fourier transform $\tilde{f}$, and the second derivative of the multiplier of t in the exponent is a multiplier independent of m and t. Two general features, independent of the initial condition, are seen from the above relation. First, the decay to the final state is not in the form of a power law, but in the form of an exponential. It becomes a power law only in the symmetric case $\eta = 0$. Second, if $\eta > 0$, the expectation at the rightmost sites tends rapidly to its final value. That is, the profile of the deviation from the final value is decreasing with

respect to m. This is so, since in this case the two-site reaction is favourable to the state where the left-hand site changes so that it becomes identical to the right-hand site. This means that cases where the right-hand site changes are less probable than cases where the left-hand site changes. So, the right-hand site arrives earlier to its final state. This expression seems to be unbounded for either $m \to \infty$ or $m \to -\infty$. For any fixed t, this is true. However, it simply means that in order that this term represents the leading term for some m, t must be greater than some T, which *does depend* on m.

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